SYMMETRY ANALYSIS OF REFRACTION DATA

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In ophthalmic geometric optics, the simplest representation of the corneal surface curvature corresponds to a spherical-cylindrical surface with the direction of the steep (maximum, $\kappa_s$) and flat (minimum, $\kappa_f$) curvatures oriented with a 90 deg angular separation. This is simply Euler Theorem of classical differential geometry. The resulting refractive profile,

$$\pi(\theta) = (\eta - \eta')[\kappa_s \cos^2(\theta - \alpha) + \kappa_f \sin^2(\theta - \alpha)], \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \alpha \leq \pi,$$

can be expressed as $\pi(\theta) = s + c \cos^2(\theta - \alpha)$, where $s = (\eta - \eta')\kappa_f$, $c = (\eta - \eta') (\kappa_s - \kappa_f)$ and $\alpha$ are respectively the spherical, cylindrical and axial (or reference angle for the $\{k_s, k_f\}$ orthogonal directions) components of the spherocylindrical corrective element$^1$, and $\eta, \eta'$ are refractive indices, e.g., Viana (2003a), Viana, Olkin and McMahon (1993). Figure 1.1 illustrates the power profiles (in polar coordinates) for $s = 4.25$, $c = -1.5$, $\alpha = 20$ deg and $s = -2.75$, $c = 1.00$, $\alpha = 10$ deg. The associated refractive power matrix,

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Refractive profile for $s = 4.25, c = -1.5, \alpha = 20$ deg (outer contour) and $s = -2.75, c = 1.00, \alpha = 10$ deg (inner contour).}
\end{figure}

using the standard Long’s notation, is given by

$$F = \begin{bmatrix} s + c \sin^2(\alpha) & -c \sin(\alpha) \cos(\alpha) \\ -c \sin(\alpha) \cos(\alpha) & s + c \cos^2(\alpha) \end{bmatrix} = \begin{bmatrix} S - C_x & -C_x \\ -C_x & S + S_x \end{bmatrix}.$$  

The RHS notation is from Campbell (1997), see also Campbell (1994), in which $S = s + c/2$, $C_x = (c/2) + \cos(2\alpha)$, $C_x = (c/2) \sin(2\alpha)$. We observe the scalars $(s, c, \alpha)$, respectively the sphere, cylinder and axis. These, in turn, form the numerical power matrix $F$, e.g.,

$$(s, c, \alpha) = (4.25, -1.5, 20 \text{ deg}) \rightarrow F = \begin{bmatrix} 4.0745 & 0.48207 \\ 0.48207 & 2.9255 \end{bmatrix}.$$  

The question to be addressed in this report is that of constructing the appropriate space within which the data $\{(s, c, \alpha), F\}$ can be properly studied. More specifically, how to construct a vector space within which

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$^1$The values of $s$, $c$ and $\alpha$ are the refractive data annotated in a typical prescription for corrective lenses.
the data operations (sum and scalar multiplication) can be meaningfully interpreted.

To address this question, the notion of data indexed by a finite group will be introduced. Given a finite group G, these are experimental data \{x(\tau); \tau \in G\} that are the scalar (typically real) coefficients defining the elements \(\sum_{\tau \in G} x(\tau)\tau\) of the group algebra \(A_G\) associated with the finite group G. Important for data analysis is the fact that the group algebra \(A_G\) can be considered as the set of all data \{x(\tau); \tau \in G\} indexed by G. It has a vector space structure given by \(y + \alpha x = \{y(\tau) + \alpha x(\tau); \tau \in G\}\) for all \(x, y \in A_G\) and scalars \(\alpha\), and a multiplicative structure defined by \(xy = \sum_{\tau, \sigma \in G} x(\tau)y(\sigma)\sigma\tau\).

In the applications discussed in this report, the group of interest is the (dihedral) group of invariant motions of a rigid regular n-sided polygon, denoted by \(D_n\). The data \{x(\tau); \tau \in D_n\} may be obtained (a) directly from experiments in which the intrinsic symmetries are those of the dihedral group or (b) indirectly as the inverse solution to a Fourier transform

\[
F = \hat{x}(\beta) = \sum_{\tau \in G} x(\tau)\beta(\tau),
\]

at an irreducible two-dimensional representation \(\beta\) of \(D_n\), given the linear operator \(F \in \mathbb{R}^{2 \times 2}\). We refer to these representations, \(\beta\), as frequencies and to the half-dimension (\(n\)) of \(D_n\) as the equation’s resolution. We will apply the Dihedral Fourier-inverse in the context of statistical geometric optics and show, as suggested in the work of Lakshminarayanan, Srudhar and Jagannathan (1998), that group theoretical analyses may allow newer methods of analysis of data consistent with the optical properties of a given problem; The role of the dihedral group is that of a probing tool with which certain symmetry experiments can be designed and the corresponding data analyzed.

In the examples to be discussed in the report we apply the inverse-Fourier formula

\[
x(\tau) = \sum_{\rho} \frac{\rho(1)}{|G|} \text{tr} [\rho(\tau^{-1})F], \quad \tau \in D_4,
\]

where the sum is over the irreducible representations of \(D_4\), to solve the equation \(F = \hat{x}(\beta)\). The resulting data are then indexed by \(D_4\), and carry the vector space structure of the group algebra of \(D_4\). Subsequent analyses of the data may then follow the steps of a standard symmetry study, as described in Viana (2003b). It leads to the analysis of the canonical projections (for the regular representation of \(D_4\)) of the data \{x(\tau); \tau \in G\} in the vector space associated with the group algebra, and to some form of analysis of variance for the inner product of interest. Moreover, it will be shown that the coefficients indexing the group algebra are exactly the coefficients

\[
C_0 = c \cos(2\alpha) = c[2(\cos(\alpha))^2 - 1], \quad C_{45} = c \sin(2\alpha) = 2c \sin(\alpha) \cos(\alpha), \quad M = \frac{s + (s + c)}{2} = s + c/2,
\]

appearing in W.E. Humphrey’s principle of astigmatic decomposition. That is, that the solution generated by the Dihedral Fourier-inverse method is exactly Humphrey’s astigmatic decomposition. In particular, the statement² that when expressed in such form, cylinders become additive follows naturally from the additive properties of the vector space (\(\mathbb{R}^8\)) defined by the underlying group algebra. It is within this vector space that statistical analyses should then be carried on.

**References**


